INITIAL CONDITIONS
FOR SEMICLASSICAL FIELD THEORY

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Abstract

Semiclassical approximation based on extracting a c-number classical component from quantum field is widely used in the quantum field theory. Semiclassical states are considered then as Gaussian wave packets in the functional Schrödinger representation and as Gaussian vectors in the Fock representation. We consider the problem of divergences and renormalization in the semiclassical field theory in the Hamiltonian formulation. Although divergences in quantum field theory are usually associated with loop Feynman graphs, divergences in the Hamiltonian approach may arise even at the tree level. For example, formally calculated probability of pair creation in the leading order of the semiclassical expansion may be divergent.This observation was interpretted as an argument for considering non-unitary evolution transformations, as well as non-equivalent representations of canonical commutation relations at different time moments. However, we show that this difficulty can be overcomed without the assumption about non-unitary evolution. We consider first the Schrodinger equation for the regularized field theory with ultraviolet and infrared cutoffs. We study the problem of making a limit to the local theory. To consider such a limit, one should impose not only the requirement on the counterterms entering to the quantum Hamiltonian but also the requirement on the initial state in the theory with cutoffs. We find such a requirement in the leading order of the semiclassical expansion and show that it is invariant under time evolution. This requirement is also presented as a condition on the quadratic form entering to the Gaussian state.

Keywords: quantum field theory, semiclassical expansion, renormalization, divergences, Schrödinger equation, complex-WKB method, pair creation, external field, canonical commutation relations, quantization.

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1. Introduction

Semiclassical approximation is of widely use in quantum field theory. Examples of applying this approximation are:

(i) qunatization in the vicinity of soliton solutions to the classical field theory [1,2,3];

(ii) investigation of processes in strong external electromagnetic and gravitational fields [4,5,6,7];

(iii) Gaussian approximation of refs. [8,9,10].

To construct semiclassical theory, one usually starts from considering the field model with action depending on fields φ and small paremeter g as follows,

$$\frac{1}{g} S[\varphi\sqrt{g}]. \tag{1}$$

Then one extracts classical part of the field which is large, of order $1/\sqrt{g}$, according to the following formula,

$$\varphi = \Phi_c/\sqrt{g} + \phi \tag{2}$$

Substituting eq.(2) to eq.(1), one finds that the leading order of the action (1) is quadratic,

$$\frac{1}{2}\phi \frac{\delta^2 S}{\delta\Phi_c \delta\Phi_c}\phi + const, \tag{3}$$

since the term which is linear in ϕ vanishes because of the classical equations of motion.

Since the field theory (3) is the theory with quadratic Lagrangian, it is exactly solvable. Quantization of the action (3) leads to the Schrödinger equation with quadratic Hamiltonian depending of the creation and annihilation operators $a^{\pm}(\mathbf{k})$ as follows,

$$\hat{H}_2 = [\frac{1}{2}a^+ \hat{A}^t a^+ + a^+ \hat{B}^t a^- + \frac{1}{2} a^- \hat{A}^{t*} a^-] + \gamma^t, \tag{4}$$

where notations like $a^{\pm}\hat{O}a^{\pm}$ are used for bilinear forms in creation and annihilation operators, $\int d\mathbf{k}d\mathbf{p}a^{\pm}(\mathbf{k})O(\mathbf{k},\mathbf{p})a^{\pm}(\mathbf{p})$, $\mathbf{k}$ and $\mathbf{p}$ are momenta of particles, the operator with the kernel $O(\mathbf{k},\mathbf{p})$ is denoted by $\hat{O}$, γ^t is a number.

Note that the coefficients A^t, B^t are time-dependent for the case of non-stationary classical field solutions,

To construct exact solutions to eq.(3), one can use the following Gaussian ansatz for the vector of the Fock space,

$$c^t \exp(\frac{1}{2}a^+ \hat{M}^t a^+)\Phi^{(0)}, \tag{5}$$

Eq. (5) obey the quadratic Schrödinger equation if

$$i\dot{\hat{M}}^t = \hat{A}^t + \hat{B}^t \hat{M}^t + \hat{M}^t (\hat{B}^t)^T + \hat{M}^t \hat{A}^{t*} \hat{M}^t. \tag{6}$$

However, in this approach the problem of divergences and renormalization in quantum field theory are not taken into account. This leads to some difficulties in the semiclassical field theory which have been studied in refs. [4,11]. Namely, expression (5) really determines

the vector of the Fock space if and only if [12] the kernel of the operator $\hat{M}^t$, $M^t(\mathbf{k},\mathbf{p})$ is square integrable,

$$\int d\mathbf{k}d\mathbf{p}|M^t(\mathbf{k},\mathbf{p})|^2 < \infty, \tag{7}$$

because the two-particle component of the vector (5) that corresponds to creation of the single pair of particles with momenta $\mathbf{k}$ and $\mathbf{p}$ is $M^t(\mathbf{k},\mathbf{p})$, so that expression (7) is the full probability of pair creation. This probability should be finite. However, evolution of the quantity (7) is prescripted by eq.(6). It was found in refs.[11,4] that for some field models like quantum gravity or QED the quantity (7) may be divergent even if it converged at the initial time moment.

This difficulty of semiclassical theory lead to the assumption that time evolution in quantum field theory is non-unitary, so that one should consider non-equivalent representations of the canonical commutation relations [11,13].

We will show in this paper that this difficulty can be overcomed in the leading order of the semiclassical expansion without the assumption about non-unitary evolution.

We will consider the renormalization procedure in the Hamoiltonian formulation. It will be shown that state vectors of the linearized quantum theory (3) are not in one-to-one correspondence with elements of the renormalized state space. This means that not the condition (7) but another condition should be imposed on the function $M^t(\mathbf{k},\mathbf{p})$. Namely, the function $M^t(\mathbf{k},\mathbf{p})$ is the probability amplitude to emit *bare* particles. However, *physical* particles are not identical to bare particles, because one should consider also the processes of self-interaction (see, for example, [14]); there is "dressing transformation" that transforms bare particles to physical particles. We will analyse this transformation and find the more complicated condition on the quadratic form entering to the Gaussian state (5) which will provide the convergence of all probabilities computated for physical particles.

This paper is organized as follows. Section 2 deals with semiclassical approximation for the field theory with ultraviolet and infrared cutoffs. Since the divergences do not arise in such a regularized theory, one can investigate the semiclassical asymptotics rigorously. We will also associate this semiclassical approach with the functional - Schrödinger - equation variational approach suggested in [8,9,10]. It will happen that these approaches are equivalent at small values of g. In section 3 we will consider the problem of making limit to the local field theory. We will study conditions on the regularized Hamiltonian and on the possible states which are necessary to make the renormalized theory finite in the leading order of the semiclassicla expansion. We will construct "dressing transformation" by using the Bogoliubov method [15,16] of variable intensity of the interaction. We will find the correct condition on M^t instead of (7). Section 4 deals with the direct analysis of eq.(6). We will find that the obtained condition on M^t is invariant under time evolution, so that it is not necessary to consider non-unitary evolution or non-equivalent representations of the canonical commutation relations at different time moments. Section 5 contains conclusion remarks and discussions of the obtained condition on M^t.

2. Semiclassical approximation for the regularized field theory

This section deals with constructing the semiclassical asymptotics in quantum field theory. For the simplicity, we consider in this paper the case of the single scalar field only,

because other cases can be considered analogously. We will study the action of the form

$$S[\varphi, g] = \int dx[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{1}{g}V_{int}(\sqrt{g}\varphi)], \qquad (8)$$

where $x = (x^0, ..., x^d)$, $V_{int}(\Phi) = O(\Phi^3)$, $\partial_\mu = \partial/\partial x^\mu$.

It is well-known that quantization of the field theory with the action (8) lead to ultraviolet and volume divergences. Therefore, one usually studies the action with ultraviolet and infrared cutoffs instead of (8). The regularized action is

$$S^0_{\Lambda,L}[\varphi, g] = \int dx[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2] - \int dx\chi(\mathbf{x}/L)\frac{1}{g}V_{int}(\sqrt{g}\varphi^\Lambda(x^0, \mathbf{x})], \qquad (9)$$

where

$$\varphi^\Lambda(t, \mathbf{x}) = \int d\mathbf{y}\rho(\Lambda(\mathbf{x} - \mathbf{y}))\varphi^\Lambda(t, \mathbf{y}),$$

χ and ρ are smooth and rapidly damping functions, $\chi(0) = 1$, $\int d\mathbf{z}\rho(\mathbf{z}) = 1$. If $\Lambda, L \to \infty$ then action (9) takes the form (8).

However, the corresponding quantum field theory is not well-defined as $\Lambda, L \to \infty$. To make this limit, $\Lambda, L \to \infty$, possible, one should add to (9) the counterterms:

$$S^0_{\Lambda,L} + S^{ct}_{\Lambda,L}, \qquad (10)$$

where the added term $S^{ct}_{\Lambda,L}(\varphi, g)$ depends on the small parameter g like this:

$$S^{ct}(\varphi, g) = \sum_{n\geq 1} g^{n-1} S^n_{\Lambda,L}(\varphi\sqrt{g})$$

. The purpose of the functional $S^n_{\Lambda,L}$ is to remove the n-loop divergences.

Let us consider quantization of the field theory with the action (10) and constructing semiclassical asymptotics in different representations.

Classical Hamiltonian corresponding to the action (10), depends on the fields $\varphi(\mathbf{x})$ and canonically conjugated momenta $\pi(\mathbf{x})$. It can be presented as a sum

$$H_{\Lambda,L} = H^0_{\Lambda,L} + H^{ct}_{\Lambda,L}$$

of the functional

$$H^0_{\Lambda,L} = \frac{1}{g}H^{(0)}_{\Lambda,L}[\pi\sqrt{g}, \varphi\sqrt{g}] = \frac{1}{g}H_0[\pi\sqrt{g}, \varphi\sqrt{g}] + \frac{1}{g}H^{int}_{\Lambda,L}[\pi\sqrt{g}, \varphi\sqrt{g}], \qquad (11)$$

where

$$H_0[\Pi(\cdot), \Phi(\cdot)] = \int d\mathbf{x}[\frac{1}{2}\Pi^2(\mathbf{x}) + \frac{1}{2}(\nabla\Phi)^2(\mathbf{x}) + \frac{m^2}{2}\Phi^2(\mathbf{x})],$$

$$H^{int}_{\Lambda,L}[\Pi(\cdot), \Phi(\cdot)] = \int d\mathbf{x}\chi(\mathbf{x}/L)V_{int}(\Phi^\Lambda(\mathbf{x})),$$

and the counterterm Hamiltonian depending on the small parameter g as

$$H^{ct}_{\Lambda,L} = \sum_{n\geq 0} g^n H^{(n+1)}_{\Lambda,L}[\pi\sqrt{g},\varphi\sqrt{g}].$$

We will define the form of $H^{(n+1)}_{\Lambda,L}$ later.

The procedure of canonical quantization is well-known. One should substitute fields φ and momenta π by the operators $\hat{\varphi}$ and $\hat{\pi}$, which act in the Hilbert state space $\mathcal{H}$, and obey the canonical commutation relations,

$$[\hat{\varphi}(\mathbf{x}),\hat{\varphi}(\mathbf{y})] = 0, [\hat{\pi}(\mathbf{x}),\hat{\pi}(\mathbf{y})] = 0, [\hat{\varphi}(\mathbf{x}),\hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \tag{12}$$

Evolution of the state is prescribed by the Schrödinger equation

$$i\frac{d}{dt}\Psi^t_{\Lambda,L,g} = \sum_{n\geq 0} g^{n-1} H^{(n)}_{\Lambda,L}[\sqrt{g}\hat{\pi}(\cdot),\sqrt{g}\hat{\varphi}(\cdot)]\Psi^t_{\Lambda,L,g}, \Psi^t_{\Lambda,L,g}\in\mathcal{H} \tag{13}$$

One usually chooses the Fock space as the space $\mathcal{H}$. Operators $\hat{\pi}$ and $\hat{\varphi}$ are presented through the creation and annihilation operators as follows

$$\begin{aligned}\hat{\pi}(\mathbf{x}) &= \tfrac{i}{(2\pi)^{d/2}}\int d\mathbf{k}\sqrt{\tfrac{\omega_{\mathbf{k}}}{2}}[a^+(\mathbf{k})e^{-i\mathbf{kx}} - a^-(\mathbf{k})e^{i\mathbf{kx}}],\\ \hat{\varphi}(\mathbf{x}) &= \tfrac{1}{(2\pi)^{d/2}}\int d\mathbf{k}\sqrt{\tfrac{1}{2\omega_{\mathbf{k}}}}[a^+(\mathbf{k})e^{-i\mathbf{kx}} + a^-(\mathbf{k})e^{i\mathbf{kx}}],\end{aligned} \tag{14}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$.

Since the operators $\hat{\pi}$ and $\hat{\varphi}$ are non-commutative according to eq.(12), there is a problem to order the operators: it is not clear how to construct operators $H^{(n)}_{\Lambda,L}[\hat{\pi}(\cdot)\sqrt{g},\hat{\varphi}(\cdot)\sqrt{g}]$, using the functionals $H^{(n)}_{\Lambda,L}[\pi(\cdot)\sqrt{g},\varphi(\cdot)\sqrt{g}]$ In quantum field theory one ususally uses the Wick ordering: the operators of fields and momenta are expressed through the creation and annihilation operators according to eq.(14), then the creation and annihilation operators are ordered in such a way that creation operators act later than annihilation operators.

Let us consider construction of semiclassical solutions to eq.(3).

In quantum mechanics, the well-known semiclassical approach is the WKB-approach, which enables one to construct semiclassical solutions to the Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H(x,-i\hbar\frac{\partial}{\partial x})\psi(x,t) \tag{15}$$

of the rapidly oscillating form

$$\psi(x,t)\sim\phi(x,t)\exp(\frac{i}{\hbar}S(x,t)), \tag{16}$$

where action S is a real function. One can easily show that the wave function (16) really obeys approximately the Schrödinger equation (15) as $\hbar\to 0$. However, the form (16) is not the only possible ansatz of the wave function, obeying approximately eq.(15) in the

semiclassical approximation, $\hbar \to 0$. One can construct other, non-WKB, semiclassical solution to (15).

Another example of semiclassical wave function is the wave-packet approximate solution to eq.(15):

$$\psi(x,t) \sim e^{\frac{i}{\hbar}S(t)} e^{\frac{i}{\hbar}P(t)(x-Q(t))} f(t, \frac{x-Q(t)}{\sqrt{\hbar}}) \tag{17}$$

corresponding to the particle moving on the classical trajectory $(P(t), Q(t))$. The wave function (17) can be considered as

$$U^{\hbar}_{P(t),Q(t),S(t)} \chi^{\hbar}, \tag{18}$$

where

$$\chi^{\hbar}(x,t) = \chi(t, x/\sqrt{\hbar}), \tag{19}$$

while $U^{\hbar}_{P,Q,S}$ is an unitary operator of the form

$$U^{\hbar}_{P,Q,S} = e^{\frac{i}{\hbar}S} e^{\frac{i}{\hbar}Px} e^{-\frac{i}{\hbar}Q(-ih\frac{\partial}{\partial x})}. \tag{20}$$

One can show by using the complex-WKB (or complex germ) technique of ref.[17] that eq.(17) really satisfies eq.(15) as $\hbar \to 0$. Wave function (17) have been also considered as a test function in the variational method of refs.[18,8,9].

It is the semiclassical wave function (17) rather than the WKB-function (15) that generalizes to the secondary-quantized systems [19,20]. Quantum-field analog of the complex-WKB method is the soliton quantization.

Let us construct quantum-field-theory analog of the asymptotics (17).

We consider the more general case of the Hamiltonian $H^{(0)}_{\Lambda,L}$ of the form

$$\frac{1}{g} H^{(0)}_{\Lambda,L}[\pi\sqrt{g}, \varphi\sqrt{g}] = \frac{1}{g} H_0[\pi\sqrt{g}, \varphi\sqrt{g}] + \xi(t)\frac{1}{g} H^{int}_{\Lambda,L}[\pi\sqrt{g}, \varphi\sqrt{g}], \tag{21}$$

where $\xi(t)$ is a smooth function of t. If $\xi(t) = 1$ then the operator (21) takes the form (11).

Note that the small parameter g is an analog of the Planck constant $\hbar$, operators $\hat{\varphi}\sqrt{g}$ and $\hat{\pi}\sqrt{g}$ are analogs of the quantum mechanical coordinate and momentum operators, x and $-i\hbar\partial/\partial x$. The g-independent state vector is an analog of the wave function (19), while the unitary transformation

$$U^{g}_{\Phi,\Pi,S} = e^{\frac{i}{g}S} e^{\frac{i}{\sqrt{g}}\int d\mathbf{x}[\Pi(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \Phi(\mathbf{x})\hat{\pi}(\mathbf{x})]},$$

where S is a real number, $\Phi(\cdot)$, $\Pi(\cdot)$ are real functions, is the analog of eq.(20). It follows from the canonical commutation relations (12) that

$$\begin{gathered} (U^{g}_{\Phi,\Pi,S})^{-1} \hat{\varphi}(\mathbf{x}) U^{g}_{\Phi,\Pi,S} = \hat{\varphi}(\mathbf{x}) + \Phi(\mathbf{x})/\sqrt{g}, \\ (U^{g}_{\Phi,\Pi,S})^{-1} \hat{\pi}(\mathbf{x}) U^{g}_{\Phi,\Pi,S} = \hat{\pi}(\mathbf{x}) + \Pi(\mathbf{x})/\sqrt{g}, \\ (U^{g}_{\Phi,\Pi,S})^{-1} \frac{d}{dt} U^{g}_{\Phi,\Pi,S} = D_t \end{gathered} \tag{22}$$

where

$$iD_t = -\frac{1}{g}\left(\dot{S}^t + \frac{1}{2}\int d\mathbf{x}(\dot{\Pi}^t(\mathbf{x})\Phi^t(\mathbf{x}) - \dot{\Phi}^t(\mathbf{x})\Pi^t(\mathbf{x}))\right)$$
$$+\frac{1}{\sqrt{g}}\int d\mathbf{x}(\dot{\Phi}^t(\mathbf{x})\hat{\pi}(\mathbf{x}) - \dot{\Pi}^t(\mathbf{x})\hat{\varphi}(\mathbf{x})) + id/dt.$$

Consider the following element of the space $\mathcal{H}$,

$$\Psi^t = U^g_{\Phi^t,\Pi^t,S^t} Y^t, \tag{23}$$

where state vector Y^t is regular as $g \to 0$. Subsituting state vector (23) to eq.(13), making use of commutation rules (22), one obtains the following equation on Y^t:

$$iD_t Y^t = \sum_{n\geq 0} g^{n-1} H^{(n)}[\Pi(\cdot) + \sqrt{g}\hat{\pi}(\cdot), \Phi(\cdot) + \sqrt{g}\hat{\varphi}(\cdot)]Y^t, \tag{24}$$

where indices Λ, L are omitted.

Eq.(24) contains singular as $g \to 0$ terms of orders $O(1/g)$ and $O(1/\sqrt{g})$. In order to obtain a regular equation for Y^t, one should make these terms equal to 0. One finds the following relation

$$\dot{S}^t = \frac{1}{2}\int d\mathbf{x}(-\dot{\Pi}^t(\mathbf{x})\Phi^t(\mathbf{x}) + \dot{\Phi}^t(\mathbf{x})\Pi^t(\mathbf{x}))) - H^{(0)}[\Pi(\cdot), \Phi(\cdot)]. \tag{25}$$

and the Hamiltonian system

$$\dot{\Phi}^t(\mathbf{x}) = \frac{\delta H^{(0)}}{\delta \Pi(\mathbf{x})}, \dot{\Pi}^t(\mathbf{x}) = -\frac{\delta H^{(0)}}{\delta \Phi(\mathbf{x})}. \tag{26}$$

Eqs.(25) and (26) imply the following equation on Y^t as $g \to 0$,

$$i\frac{d}{dt}Y^t = H_2[\hat{\pi}(\cdot), \hat{\varphi}(\cdot)]Y^t, \tag{27}$$

where quantum operator $\hat{H}_2 = H_2[\hat{\pi}(\cdot), \hat{\varphi}(\cdot)]$ corresponds to the classical functional

$$H_2[\pi(\cdot), \varphi(\cdot)] = \frac{1}{2}\pi\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi}\pi + \pi\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Phi}\varphi + \frac{1}{2}\varphi\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}\varphi + H^{(1)}. \tag{28}$$

the Wick ordering is assumed for $H_2[\hat{\pi}(\cdot), \hat{\varphi}(\cdot)]$. In eq.(28) the arguments $\Phi(\cdot), \Pi(\cdot)$ of the functionals $H^{(0)}$ and $H^{(1)}$ are omitted, while integrals like $\int \pi(\mathbf{x})\frac{\delta^2 H^{(0)}}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{y})}\pi(\mathbf{y})d\mathbf{x}d\mathbf{y}$ are denoted as $\pi\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi}\pi$.

Note that an alternative way to derive the Schrödinger equation (27) for the linearized field theory is to extract the classical part of the field according to eq.(2) and to quantize the Lagrangian (3) canonically.

Let us consider exact solutions to eq.(27) being of the type (4), where

$$\hat{A}^t(\mathbf{k},\mathbf{p}) = \int \frac{d\mathbf{x}d\mathbf{y}}{(2\pi)^d} e^{-i\mathbf{kx}} \frac{1}{4} \left(\frac{1}{\sqrt{\omega_{\mathbf{k}}}} \frac{\delta^2 H^{(0)}}{\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{y})} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \right.$$

$$\left. -\sqrt{\omega_{\mathbf{k}}} \frac{\delta^2 H^{(0)}}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{y})} \sqrt{\omega_{\mathbf{k}}} \right) e^{-i\mathbf{py}},$$

$$\hat{B}^t(\mathbf{k},\mathbf{p}) = \int \frac{d\mathbf{x}d\mathbf{y}}{(2\pi)^d} e^{-i\mathbf{kx}} \frac{1}{4} \left(\frac{1}{\sqrt{\omega_{\mathbf{k}}}} \frac{\delta^2 H^{(0)}}{\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{y})} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \right.$$

$$\left. +\sqrt{\omega_{\mathbf{k}}} \frac{\delta^2 H^{(0)}}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{y})} \sqrt{\omega_{\mathbf{k}}} \right) e^{i\mathbf{py}}, \tag{29}$$

while $\gamma^t = H^{(1)}$. We have taken into account that $\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Phi} = 0$.

Eq.(4) hash solutions of the form (5), where (6) is satisfied, while

$$i\dot{c}^t = (Tr\hat{A}^t\hat{M}^t + H^{(1)})c^t. \tag{30}$$

To construct other exact solutions to eq.(4) which obey initial conditions of the type

$$P(a^+)\exp(\frac{1}{2}a^+\hat{M}^0 a^+)\Phi^{(0)},$$

where $P(a^+)$ is a polynomial in creation operators, one can introduce the so-called complex germ (or complex-WKB) creation operators of ref.[17]:

$$\Lambda[p^t, q^t] = \int d\mathbf{x}[p^t(\mathbf{x})\hat{\varphi}(\mathbf{x}) - q^t(\mathbf{x})\hat{\pi}(\mathbf{x})] \tag{31}$$

Because of the canonical commutation relations

$$[i\frac{d}{dt} - \hat{H}_2, \Lambda[p^t,q^t]] = i\Lambda[\dot{p}^t + \frac{\delta H_2}{\delta q}, \dot{q}^t - \frac{\delta H_2}{\delta p}] \tag{32}$$

operator (31) transforms a solution to eq.(27) into a solution if

$$\dot{p}^t(\vec{x}) = -\frac{\delta H_2(p^t(\cdot), q^t(\cdot))}{\delta q(\mathbf{x})}, \dot{q}^t(\vec{x}) = \frac{\delta H_2(p^t(\cdot), q^t(\cdot))}{\delta p(\mathbf{x})}. \tag{33}$$

Let us simplify eqs.(6) and (30). Consider the operator $\tilde{M}^t$ with the kernel being the Fourier transformation of the kernel $M^t(\mathbf{k},\mathbf{p})$ of the operator $\hat{M}^t$,

$$(\tilde{M}^t f)(\mathbf{x}) = \int \frac{d\mathbf{k}d\mathbf{p}d\mathbf{y}}{(2\pi)^d} e^{i(\mathbf{kx}+\mathbf{py})} M^t(\mathbf{k},\mathbf{p}) f(\mathbf{y}), \tag{34}$$

Making use of this substitution (34), one finds that

$$i\dot{\tilde{M}}^t = \frac{1}{2}(1+\tilde{M}^t)\frac{1}{\sqrt{\hat{\omega}}}\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}\frac{1}{\sqrt{\hat{\omega}}}(1+\tilde{M}^t) - \frac{1}{2}(1-\tilde{M}^t)\sqrt{\hat{\omega}}\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi}\sqrt{\hat{\omega}}(1-\tilde{M}^t) \quad (35)$$

where $\hat{\omega}$ is the operator $\hat{\omega} = \sqrt{-\Delta + m^2}$. while $\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}$ and $\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi}$ are operators with the kernels $\frac{\delta^2 H^{(0)}}{\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{y})}$ and $\frac{\delta^2 H^{(0)}}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{y})}$ correspondingly. Eqs.(30) can be written as

$$(lnc^t)^{\cdot} = -\frac{i}{4}[(\frac{1}{\sqrt{\hat{\omega}}}\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}\frac{1}{\sqrt{\hat{\omega}}} - \sqrt{\hat{\omega}}\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi}\sqrt{\hat{\omega}})\tilde{M}^t] - iH^{(1)}. \quad (36)$$

Equation for the quadratic form entering to the Gaussian state vector, can be simplified if one considers the functional Schrödinger representation of refs.[8,9] instead of the Fock representation. In the functional representation, states are presented as functionals $\Psi[\phi(\cdot)]$ of a real function $\phi(\mathbf{x})$, operators $\hat{\varphi}(\mathbf{x})$ are presented as operators of multiplication by $\phi(\mathbf{x})$, operators $\pi(\mathbf{x})$ are differential operators, $-i\delta/\delta\phi(\mathbf{x})$.

Let us transform the Gaussian vector (5) of the Fock space into the Schrödinger representation. Notice that the vector (5) is determined uniquely from the relation

$$(a^-(\mathbf{k}) - (\hat{M}^t a^+)(\mathbf{k}))Y = 0. \quad (37)$$

which can be easily transformed into the Schrödinger representation, because the creation and annihilation operators can be expressed through the field and momenta operators. Namely, the operators

$$\tilde{a}^{\pm}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{d/2}} a^{\pm}(\mathbf{k}) e^{\mp i\mathbf{k}\mathbf{x}},$$

are expressed as follows,

$$\tilde{a}^{\pm} = \sqrt{\frac{\hat{\omega}}{2}}\hat{\varphi} \mp i\frac{1}{\sqrt{2\hat{\omega}}}\hat{\pi}.$$

so that eq.(37) can be presented as

$$\left[(1-\tilde{M})\sqrt{\frac{\hat{\omega}}{2}}\phi - (1+\tilde{M})\frac{1}{\sqrt{2\hat{\omega}}}\frac{\delta}{\delta\phi}\right] Y[\phi(\cdot)] = 0.$$

The solution to this equation has the Gaussian form,

$$Y[\phi(\cdot)] = c_q^t \exp\left[\frac{i}{2}\int d\mathbf{x}\phi(\mathbf{x})(R\phi)(\mathbf{x})\right] \quad (38)$$

where

$$R = i\sqrt{\hat{\omega}}(1-\tilde{M})(1+\tilde{M})^{-1}\sqrt{\hat{\omega}} \quad (39)$$

These Gaussian functionals were considered as test functionals of the variational method of [8,9].

The equation for the quadratic form takes the form

$$\dot{R} + R^2 + \frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi} = 0, \tag{40}$$

where the relation $\frac{\delta^2 H^{(0)}}{\delta\Pi\delta\Pi} = 1$ is taken into account. Making use of the following transformation of the pre-exponential factor,

$$c_q^t = \frac{c^t}{\sqrt{det(1+\tilde{M})}} \tag{41}$$

one finds from eq.(30) that

$$(lnc_q^t)^{\cdot} = -\frac{1}{2}Tr(R - i\sqrt{-\Delta+m^2} - \frac{i}{2\sqrt{-\Delta+m^2}}\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}) - iH^{(1)}. \tag{42}$$

Note that eqs.(40),(42) can be found by direct substituting of the functional (38) to eq.(27). The relation between pre-exponential factors in eqs. (5) and (38) can be obtained by expressing the vector (5) through the coherent states according to [12] and using the well-known formulas for the coherent state in the functional representation.

Let us substract the argument of the complex factor c_q^t. Consider the real and imaginary parts of the operator R, which are Hermitian. Since the imaginary part of R satisfies the equation $(ImR)^{\cdot} = -ReRImR - ImRReR$, one has $(lndetImR)^{\cdot} = -2TrReR$. Thus, the quantity

$$a^t = c_q^t/(detImR)^{1/4} = \frac{c^t}{(det(1+\tilde{M}))^{1/2}(detImR)^{1/4}} \tag{43}$$

satisfies the equaton

$$i(lna^t)^{\cdot} = -\frac{1}{2}Tr(ImR - \sqrt{-\Delta+m^2} - \frac{1}{2\sqrt{-\Delta+m^2}}\frac{\delta^2 H^{(0)}}{\delta\Phi\delta\Phi}) - H^{(1)}. \tag{44}$$

3. The problem of renormalization in the Hamiltonian formulation

Let us consider the semiclassical formulas in the limit $\Lambda \to \infty$, $L \to \infty$. This problem is non-trivial because the evolution operator is singular in this limit, contrary to the S-matrix: formal Schrödinger equation in the local field theory is not well-defined because of the Stueckelberg divergences [21].

In the constructive field theory (see, for example, [22,23,24,25]) this limit is considered as follows. One considers the "renormalized" state space $\mathcal{H}_{ren}$ and unitary operator $T_{\Lambda,L} : \mathcal{H}_{ren} \to \mathcal{H}$, depending on the parameters of the ultraviolet and infrared cutoffs. Let $U^t_{\Lambda,L}$ be the evolution operator in the regularized theory. The operator $T_{\Lambda,L}$ which is usually called as "dressing transformation" is shoosen in order to make the operators

$$W^t_{\Lambda,L} = (T_{\Lambda,L})^{-1}U^t_{\Lambda,L}T_{\Lambda,L} \tag{45}$$

regular as $\Lambda, L \to \infty$. The operator $\lim_{\Lambda,L\to\infty} W^t_{\Lambda,L}$ is the renormalized evolution operator.

This definition means the following. If the initial condition for eq.(13) is of the form

$$T_{\Lambda,L}X, \tag{46}$$

where X is Λ, L-independent renormalized state vector from $\mathcal{H}_{ren}$, then the solution to the Schrödinger equation is written in analogous way,

$$T_{\Lambda,L}X^t + Z^t_{\Lambda,L},$$

where $Z^t_{\Lambda,L}$ tends to zero as $\Lambda, L \to \infty$.

Note that if the operator expressions $(T_{\Lambda,L})^{-1}a^{\pm}(\mathbf{k})T_{\Lambda,L}$ are regular as $\Lambda, L \to \infty$ then the transformation $T_{\Lambda,L}$ allows us to construct a representation of the canonical commutation relations. This representation may be of the Fock or of the non-Fock type.

However, the transformations $T_{\Lambda,L}$ have been constructed beyond perturbation theory only for some simple cases, namely, for two-dimensional and three-dimensional models [22,23,24,25]. For general quantum field models the operator $T_{\Lambda,L}$ can be constructed only as a formal perturbation series in $g^{1/2}$ (examples are cited in refs.[14,23,26]). One can prove then that the coefficients of the expansion of the operator (45) are regular.

Let us define the notion of the semiclassically-regular operator (45). We will say that the operator is semiclassically regular if the parameters entering to the constructed semiclassical asymptotics of the vector $W^t_{\Lambda,L}\Psi^g_{\Lambda,L}$ have limits as $\Lambda, L \to \infty$. This means that for all pairs of smooth and rapidly damping functions $\Phi^0(\mathbf{x})$, $\Pi^0(\mathbf{x})$, there exists the pair of functions $\Phi^0_{\Lambda,L}(\mathbf{x}), \Pi^0_{\Lambda,L}(\mathbf{x})$ and number S^t that the operator

$$V^t_{\Lambda,L,g} = (U^g_{\Phi^t_{\Lambda,L},\Pi^t_{\Lambda,L},S^t_{\Lambda,L}})^{-1} W^t_{\Lambda,L,g} U^g_{\Phi^0,\Pi^0,0} \tag{47}$$

has a limit as $g \to 0$, while the operators

$$\lim_{g\to 0}(U^g_{\Phi^t_{\lambda,L},\Pi^t_{\Lambda,L},S^t_{\Lambda,L}})^{-1} W^t_{\Lambda,L,g} U^g_{\Phi^0,\Pi^0,0} \tag{48}$$

are regular as $\Lambda, L \to \infty$. We will call the operator $T_{\Lambda,L}$ entering to eq.(45) as the "semiclassical dressing transformation".

Let us construct such an operator, making use of the Bogoliubov method of variable intensity of the interaction [15,16] (see also [27]).

Consider the smooth function $\xi(\tau)$, $\tau \le 0$ vanishing as $\tau < -T_1$ and being equal to 1 at $\tau > -T_2$ (fig.1). Consider the operator

$$H^{\Lambda,L}_{int} = \xi(\tau)\int d\mathbf{x}\chi(\mathbf{x}/L)\frac{1}{g}V_{int}(\sqrt{g}\varphi^{\Lambda}(\mathbf{x})) + H_{ct}(\sqrt{g}\hat{\varphi}(\cdot), \sqrt{g}\hat{\pi}(\cdot), \tau, \xi(\cdot)) \tag{49}$$

When $\tau \in (-T_2, 0)$, the functional H_{ct} is τ-independent, while $H_{ct} = 0$ as $\tau < -T_1$ The counterterm H_{ct} is to be constructed in the next section.

Let $H_0 = \int d\mathbf{k}\omega_{\mathbf{k}} a^+(\mathbf{k})a^-(\mathbf{k})$ be a free Hamiltonian. Consider the operator

$$T^{\Lambda,L} = T\exp[-i\int_{-\infty}^{0} d\tau e^{iH_0\tau} H_{int}^{\Lambda,L}(\tau) e^{-iH_0\tau}] \tag{50}$$

which transforms the initial condition of the Cauchy problem

$$\begin{gathered} i\frac{dX^\tau}{d\tau} = e^{iH_0\tau} H_{int}^{\Lambda,L}(\tau) e^{-iH_0\tau} X^\tau \\ X^{-\infty} = X \end{gathered} \tag{51}$$

to the solution to this problem as $\tau = 0$. The right-hand side of eq.(51) vanish at $\tau < -T_1$, so that the initial condition at $\tau = -\infty$ can be substituted by the initial condition at $\tau = -T_0 < -T_1$.

Consider the semiclassical approximation for the vector (46).

Introduce the following notation. Let S and lna be real numbers, $\Phi(\mathbf{x})$, $\Pi(\mathbf{x})$, $q(\mathbf{x})$, $p(\mathbf{x})$ be smooth and rapidly damping at the infinity functions, R be an operator with symmetric kernel and positively defined imaginary part. Consider these objects as initial conditions for the dynamical equations (26), (25), (44), (33), (40) at $t = t_1$:

$$p^{t_1} = p, q^{t_1} = q, \Phi^{t_1} = \Phi, \Pi^{t_1} = \Pi, R^{t_1} = R, S^{t_1} = S, a^{t_1} = a. \tag{52}$$

Consider the solutions of these equations at $t = t_2$:

$$p^{t_2}, q^{t_2}, \Phi^{t_2}, \Pi^{t_2}, R^{t_2}, S^{t_2}, a^{t_2}. \tag{53}$$

Denote this transformation as $\mathcal{U}^{t_2,t_1}_{\xi(\cdot)}$:

$$(p^{t_2}, q^{t_2}, \Phi^{t_2}, \Pi^{t_2}, R^{t_2}, S^{t_2}, a^{t_2}) = \mathcal{U}^{t_2,t_1}_{\xi(\cdot)}(p^{t_1}, q^{t_1}, \Phi^{t_1}, \Pi^{t_1}, R^{t_1}, S^{t_1}, a^{t_1})$$

where $\xi(\cdot)$ enter to eq.(21).

To construct state $T^{\Lambda,L}X^{-\infty}$ as $g \to 0$, consider the substitution $\Psi^t = e^{-iH_0\tau}X^\tau$ for eq.(51). The Cauchy problem (51) takes the form

$$\begin{gathered} i\frac{d\Psi^\tau}{d\tau} = [H_0 + H_{int}^{\Lambda,L}(\tau)]\Psi^\tau \\ \Psi^{-T_0} = e^{iH_0T_0}X^{-\infty} \end{gathered} \tag{54}$$

Let us consider the initial condition $X^{-\infty}$

$$\begin{gathered} X_g^{-\infty} = U^g_{\Phi,\Pi,0} c\Lambda[p_1(\cdot), q_1(\cdot)]...\Lambda[p_k(\cdot), q_k(\cdot)] \\ \times \exp(\tfrac{1}{2}\int d\mathbf{k} a^+(\mathbf{k})(Ma^+)(\mathbf{k}))\Phi^{(0)}, \end{gathered} \tag{55}$$

Φ, Π, c, M are regular as $\Lambda, L \to \infty$, $||M|| < 1$, and the property (7) is satisfied.

According to the previous section, asymptotic solution to the Cauchy problem (54) at $\tau = 0$ is

$$\begin{gathered} \Psi_g^0 = U^g_{\Phi',\Pi',S'} c'\Lambda[p'_1(\cdot), q'_1(\cdot)]...\Lambda[p'_k(\cdot), q'_k(\cdot)] \\ \exp(\tfrac{1}{2}\int d\mathbf{k} a^+(\mathbf{k})(M'a^+)(\mathbf{k}))\Phi^{(0)}, \end{gathered} \tag{56}$$

where the parameters entering to eq. (56), have the form

$$(p'_j, q'_j, \Phi', \Pi', R(M'), S', a(c')) = \mathcal{U}^{0,-T_0}_{\xi(\cdot)} \mathcal{U}_0^{-T_0,0}(p_j, q_j, \Phi, \Pi, R(M), 0, a(c)), \quad j = \overline{1,k},$$

$R(M)$ is the operator (39), and $a(c)$ has the form (43).

Since eqs. (25), (26), (33) are regular as $\Lambda, L \to \infty$, parameters $\Phi', \Pi', S', p'_j, q'_j$ are also finite in this limit.

However, the integral (7) can be divergent in this limit, even if it was finite at

The operator M' is a solution to eq. (6), which is equal to M^{-T_0} at the initial time moment $t = -T_0$. Remind that the integral (7) should be convergent at $t = -T_0$ even if one makes a limit $\Lambda, L \to \infty$. However, the integral (7) can be divergent in this limit at $t = 0$.

Thus, we see that the Gaussian vector (23) of the regularized theory

$$u^g_{\Phi,\Pi,0} const \exp(\frac{1}{2}\int d\mathbf{k} a^+(\mathbf{k})(Ma^+)(\mathbf{k}))\Phi^{(0)} \tag{57}$$

correspond to renormalized state vector not if the quantity (7) is finite for the operator M, but if it is finite at $t < -T_0$ for the solution to eq.(35), which is equal to $\tilde{M}$ at $t = 0$. The pre-factor of (57) is singular as $\Lambda, L \to \infty$; the dependence on Λ, L is determined from the following condition: the solution to eq. (36) which coincides with c^0 at $t = 0$ is regular at $t < -T_1$ as $\Lambda, L \to \infty$.

Let us reformulate this statement for the operator R. Consider the operator $\tilde{M}$ as a function of the coordinate and differential operators: $\tilde{M} = \tilde{M}(\overset{2}{\mathbf{x}}, -i\overset{1}{\partial}/\partial\mathbf{x})$. The function $\tilde{M}(\mathbf{x}, \mathbf{k})$ is called as a symbol of the operator $\tilde{M}$. If the behaviour of the symbol at the infinity is the following

$$\tilde{M}(\mathbf{x}, \mathbf{k}) = O(|\mathbf{k}|^{-d/2-\delta}).$$

then the property (7) is satisfied as $\delta > 0$ and not satisfied as $\delta \le 0$. It follows from the formula (39) that the behavour of the symbol of the operator R at the infinity is

$$R(\mathbf{x}, \mathbf{k}) = i\sqrt{\mathbf{k}^2 + m^2} + O(|\mathbf{k}|^{-d/2-\delta+1}) \tag{58}$$

At $\Lambda, L \to \infty$, the classical Hamiltonian system (26) takes the form

$$\ddot{\Phi}_c - \Delta\Phi_c + m^2\Phi_c + \xi(t)V'_{int}(\Phi_c) = 0 \tag{59}$$

while eq.(40) takes the form,

$$\dot{R} + R^2 + (-\Delta + m^2 + \xi(t)V''_{int}(\Phi_c(t, \mathbf{x})) = 0 \tag{60}$$

Let $\Phi(t, \mathbf{x})$ be a solution to eq.(59) which satisfies the initial condition $\Phi_c|_{t=0} = \Phi$, $\dot{\Phi}_c|_{t=0} = \Pi$. Consider the state vector of the regularized theory (57). It corresponds to the following functional $\Phi[\phi(\cdot)]$ in the Schrödinger representation,

$$\begin{gathered}\Phi[\phi(\cdot)] = const \exp[\tfrac{i}{g}\Pi(\mathbf{x})(\phi(\mathbf{x})\sqrt{g} - \Phi(\mathbf{x})) \\ +\tfrac{i}{2}\int d\mathbf{x}(\phi(\mathbf{x}) - \tfrac{\Phi(\mathbf{x})}{\sqrt{g}})(R(\phi - \tfrac{\Phi}{\sqrt{g}}))(\mathbf{x}]\end{gathered} \tag{61}$$

The obtained result can be formulated as follows.

Statement 1. *The Gaussian functional (61) corresponds to the renormalized state, if the solution to eq.(60), which is equal to* R *at* $t=0$*, satisfies at* $t<-T_1$ *the condition (58) for* $\delta>0$*. If the property (58) is satisfied for* $\delta>0$ *at* $t<-T_1$*, then the functional (61) does not correspond to any state vector of the renormalized state space.*

In the next sextion we will analyse eq.(60) and formulate this condition in the more convenient form.

Let us consider the problem of regularity of the operator (48) as $\Lambda, L\to\infty$. Any vector of the Fock space can be approximated by the linear combinations of vectors

$$Y = c\Lambda[p_1(\cdot), q_1(\cdot)]...\Lambda[p_k(\cdot), q_k(\cdot)]\Phi^{(0)}, \tag{62}$$

Consider the vector (62) and analyse the problem of regularity of the vector

$$\lim_{g\to 0} V^t_{\Lambda,L,g} Y \tag{63}$$

at $\Lambda, L\to\infty$. Semiclassical asymptotics of the vector

$$W^t_{\Lambda,L,g}U^g_{\Phi^0,\Pi^0,0}Y = (T_{\Lambda,L})^{-1}U^t_{\Lambda,L}T_{\Lambda,L}U^g_{\Phi^0,\Pi^0,0}Y$$

is constructed according to the previous section. It has the form $U^g_{\Phi',\Pi',S'}Y'$, where

$$Y' = c'\Lambda[p'_1(\cdot), q'_1(\cdot)]...\Lambda[p'_k(\cdot), q'_k(\cdot)]\exp(\frac{1}{2}\int d\mathbf{k}a^+(\mathbf{k})(M'a^+)(\mathbf{k}))\Phi^{(0)}, \tag{64}$$

The parameters entering to eq.(64) are

$$\begin{gathered}(p'_j, q'_j, \Phi', \Pi', R(M'), S', a(c')) = \\ \mathcal{U}_0^{0,-T_0}\mathcal{U}_{\xi(\cdot)}^{-T_0,0}\mathcal{U}_{\xi(\cdot)}^{t,-T_0}\mathcal{U}_0^{-T_0,0}(p_j, q_j, \Phi^0, \Pi^0, R(0), 0, a(c)), j=\overline{1,k},\end{gathered} \tag{65}$$

the function $\xi(t)$ is continued at $t>0$ as $\xi(t)=1$. If $\Phi^t_{\lambda,L}=\Phi'$, $\Pi^t_{\Lambda,L}=\Pi'$, $S^t_{\Lambda,L}=S'$, then the vector (63) coincides with the vector (64).

Since p'_j, q'_j are regular as $\Lambda, L\to\infty$, it is sufficient to show that

(i) for M' the integral (7) is finite;

(ii) c' is regular as $\Lambda, L\to\infty$.

Note that the requirement (ii) is equivalent to the requirement that the quantity $lna' = lna(c')$ is finite if the condition (i) is satisfied. The requirement (i) means that for the symbol of the operator $R' = R(M')$ the property (58) is satisfied.

Let us show that the conditions (i) and (ii) really take place, if one chooses the counterterm $H^{(1)}$ properly.

4. Singularities of the quadratic form

Consider (60). Denote by $R(\mathbf{x},\mathbf{k})$ the symbol of the operator R:

$$R = R(\overset{2}{\mathbf{x}}, -i\overset{1}{\frac{\partial}{\partial\mathbf{x}}}).$$

Indices 1 and 2 mean that the differential operators act first and multiplication operators act next:

$$R \int d\mathbf{k} f(\mathbf{k}) e^{i\mathbf{kx}} \equiv \int d\mathbf{k} R(\mathbf{x},\mathbf{k}) f(\mathbf{k}) e^{i\mathbf{kx}}. \qquad (66)$$

In this section we investigate the asymptotics of the symbol $R(\mathbf{x},\mathbf{k})$ at $\mathbf{k}\to\infty$.

We consider the asymptotic expansion of the sumbol in $1/\omega_{\mathbf{k}}$, $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$ at $\mathbf{k}\to\infty$:

$$R^t(\mathbf{x},\mathbf{k}) = \mathcal{R}^t(\mathbf{x},\mathbf{k}) = i\omega_{\mathbf{k}} + \sum_{m\ge 1} \frac{R^t_m(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}})}{\omega^m_{\mathbf{k}}}, \qquad (67)$$

where $R^t_m(\mathbf{x},\mathbf{n})$ are smooth functions, rapidly damping at $\mathbf{x}\to\infty$ with all their derivatives. For the case of $\mathbf{x}$- independent fields, such an expansion was considered in ref.[9].

Let us substitute eq.(67) to eq.(60) and find the conditions which are necessary to make the left-hand side of eq.(60) of order $O(\frac{1}{\omega^j_{\mathbf{k}}})$, where j – is an arbitrary number.

Let us use the formula for the symbol of the product of the operators (see, for example, [28,29]). Let $\hat{A} = A(\overset{2}{\mathbf{x}}, -i\overset{1}{\frac{\partial}{\partial\mathbf{x}}})$, $\hat{B} = B(\overset{2}{\mathbf{x}}, -i\overset{1}{\frac{\partial}{\partial\mathbf{x}}})$ be operators. Then their product $\hat{A}\hat{B}$ has the following symbol $A*B$,

$$\hat{A}\hat{B} = A*B(\overset{2}{\mathbf{x}}, -i\overset{1}{\frac{\partial}{\partial\mathbf{x}}}),$$

which is presented as follows:

$$(A*B)(\mathbf{x},\mathbf{k}) = A(\overset{2}{\mathbf{x}}, \mathbf{k} - i\overset{1}{\frac{\partial}{\partial\mathbf{x}}}) B(\mathbf{x},\mathbf{k}). \qquad (68)$$

$$(A*B)(\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{z}d\mathbf{p}}{(2\pi)^d} A(\mathbf{x},\mathbf{k}-\mathbf{p}) B(\mathbf{x}+\mathbf{z},\mathbf{k}) e^{i\mathbf{pz}} \qquad (69)$$

To obtain eq.(69), it is sufficient to make use of the relation

$$\hat{B}\int d\mathbf{p} f(\mathbf{p}) e^{i\mathbf{px}} = \int d\mathbf{k} e^{i\mathbf{kx}} \int \frac{d\mathbf{z}d\mathbf{p}}{(2\pi)^d} e^{i(\mathbf{p}-\mathbf{k})\mathbf{y}} B(\mathbf{y},\mathbf{p}) f(\mathbf{p}),$$

and of the formula (66). Formula (68) is a corollary of eq.(69).

Perform the formal expansion of the right-hand side of the formula (68). One obtains:

$$(A*B)(\mathbf{x},\mathbf{k}) = \sum_{l\ge 0} \frac{(-i)^l}{l!} \frac{\partial^l A(\mathbf{x},\mathbf{k})}{\partial k_{i_1}...\partial k_{i_l}} \frac{\partial^l B(\mathbf{x},\mathbf{k})}{\partial x_{i_1}...\partial x_{i_l}}, \qquad (70)$$

where we sum over repeated indices. If A and B are

$$A = \frac{A^t_1(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}})}{\omega^{m_1}_{\mathbf{k}}}, B = \frac{B^t_1(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}})}{\omega^{m_2}_{\mathbf{k}}},$$

the series (70) is an asymptotic series in terms of $\frac{1}{\omega_{\mathbf{k}}}$, because the l-th term of formula (70) is of order $O(\omega_{\mathbf{k}}^{-m_1-m_2-l})$ at $\mathbf{k}\to\infty$. One can prove eq. (70) by applying the stationary phase method [30] to the integral (69).

Making use of formula (70), substituting the symbol $R*R$ of the operator R^2 to eq.(60) and considering the terms of order $O(1/\omega_{\mathbf{k}}^s)$, $s\geq 0$, we find:

$$2iR_1^t(\mathbf{x},\mathbf{n})+\xi(t)V''_{int}(\Phi^t(\mathbf{x}))=0$$

at $s=0$ and

$$\dot{R}_s^t(\mathbf{x},\mathbf{n})+2iR_{s+1}^t(\mathbf{x},\mathbf{n})+\sum_{\substack{l+m=s+1\\ l,m\geq 1}}\omega_{\mathbf{k}}^{l-1}\frac{(-i)^{l-1}}{l!}\frac{\partial^l\omega_{\mathbf{k}}}{\partial k_{i_1}...\partial k_{i_l}}\frac{\partial^l R_m^t(\mathbf{x},\mathbf{n})}{\partial x_{i_1}...\partial x_{i_l}},$$

$$+\sum_{\substack{m+n+l=s\\ m,n\geq 1,l\geq 0}}\frac{(-i)^l}{l!}\omega_{\mathbf{k}}^{m+l}\frac{\partial^l}{\partial k_{i_1}...\partial k_{i_l}}\frac{R_m^t(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^m}\frac{\partial^l R_n^t(\mathbf{x},\mathbf{n})}{\partial x_{i_1}...\partial x_{i_l}}=0, \tag{71}$$

at $s\geq 1$; in this formula $\mathbf{n}=\mathbf{k}/\omega_{\mathbf{k}}$. We have used the assumption that quantities $\dot{R}_s^t$ and R_s^t are of the same order.

Recursive relation (71) allows us to express R_{s+1}^t through the previous orders of the perturbation expansion in $1/\omega_{\mathbf{k}}$, $R_1^t,...,R_s^t$. Note that the constructed function R_{s+1}^t is determined uniquely if one knows the values of the field $\Phi^t(\cdot)$ and momenta $\Pi^t(\cdot)$ at this time moment t and values of the function ξ and its derivatives at time t.

If $\xi(t)=0$, .. $t<-T_1$, the obtained coefficients are also equal to zero, so that $R^t=i\omega_{\mathbf{k}}$ at these values of t. If $t>-T_2$ the asymptotics is ξ-independent.

The first functions R_s^t defined from eqs. (71) have the form:

$$R_1^t(\mathbf{x},\mathbf{n})=\frac{i}{2}\xi(t)V''_{int}(\Phi^t(\mathbf{x})),$$

$$R_2^t(\mathbf{x},\mathbf{n})=-\frac{1}{4}\left(\frac{\partial}{\partial t}+\mathbf{n}\frac{\partial}{\partial\mathbf{x}}\right)\xi(t)V''_{int}(\Phi^t(\mathbf{x})).$$

Note that the first derivative of Φ^t is expressed through the momentum Π^t, while next derivatives entering to next functions R_s, can be expressed from classical equations of motion through Φ^t and Π^t.

We have constructed the asymptotics at $|\mathbf{k}|\to\infty$ only for one solution to eq.(60) corresponding to vacuum at $t<-T_1$, because it is equal at these values of t to $i\omega_k$. Consider the construction of the asymptotic solution to eq.(60) which obeyes the initial condition (58). Consider the following substitution to eq.(60):

$$R^t=\mathcal{R}^t+r^t,$$

where $\mathcal{R}^t$ is the constructed solution (67). Considering the leading order in $1/\omega_{\mathbf{k}}$, one obtains that:

$$\dot{r}^t+2i\omega_{\mathbf{k}}r^t=0.$$

The solution to this equation is $r^T = r^{-T_0}\exp(-2i\omega_{\mathbf{k}}(t+T_0))$; the corrections are constructed in [31].

Thus, statement 1 can be reformulated as follows.

Statement 2. *The Gaussian functional (61) correspond to the renormalized state if and only if R^0 is equal to $\mathcal{R}^0$ (eq.(67)) up to $O(|k|^{-d/2+1-\delta})$*

$$R^0 = \mathcal{R}^0 + O(|k|^{-d/2+1-\delta}), \tag{72}$$

The requirement (72) does not depend on choice of the function $\xi(t)$ and depends on the functions $\Pi^0(\mathbf{x})$ and $\Phi^0(\mathbf{x})$ only.

Let us check the conditions (i) and (ii) formulated for R' and a' at the end of the previous section.

The condition on R' is a corollary of the property of invariance of the asymptotics (67) under substituting the function $\xi(t)$ by the function $\xi(t-t_0)$, where $t_0 = const.$

The quantity a' is finite if the counterterm $H^{(1)}$ compencate the divergences arising when one calculates the trace of R. To find the form of the counterterm, one should consider asymptotic solutions of the regularized equation (40):

$$\mathcal{R}^t_{\Lambda,L}(\mathbf{x},\mathbf{k}) = i\omega_{\mathbf{k}} + \sum_{m\ge 1, j\ge 0} \frac{R^t_{mj}(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}},\mathbf{k}/\Lambda)}{\omega^m_{\mathbf{k}}\Lambda^j},$$

extract the divergence in the trace

$$Tr(Im\mathcal{R}^t_{\Lambda,L} - i\omega_k) \sim \sum_{m=\overline{1,d}, j=\overline{0,d}} \int \frac{d\mathbf{k}d\mathbf{x} Im R^t_{mj}(\mathbf{x},\mathbf{k}/\omega_{\mathbf{k}},\mathbf{k}/\Lambda)}{\omega^m_{\mathbf{k}}\Lambda^j},$$

add it to the divergence part of the trace $\frac{1}{2}Tr\frac{1}{\tilde{\omega}}\frac{\partial^2 H^{(0)}}{\partial\Phi\partial\Phi}$ and consider $H^{(1)}$ to be equal to this sum. Since R^t_{mj} depends only on $\Phi^t(\cdot)$, $\Pi^t(\cdot)$ and on values of the function ξ and its derivatives at time moment t, the counterm $H^{(1)}$ is a functional of the field and of the momenta at $t > -T_2$. Thus, we have checked the requirements (i) and (ii).

5. Conclusion

Thus, we have found the condition formulated in statements 1 and 2 instead of eq.(7). If this condition is satisfied, the Gaussian functional (61) corresponds to an element of the renormalized state space.

Let us compare the obtained condition on the quadratic form entering to the Gaussian vector with the prescriptions formulated in other papers on the semiclassical field theory. It is usually non-manifestly assumed in the papers that do not contain the analysis of renormalization that the renormalized states coincide with states in the regularized field theory, so that one should use the condition (7). This means that the requirement (58) should be imposed on R. However, this condition may be non-invariant under time evolution.

It was suggested in ref.[32] that one should consider initial states which are eigenstates for the Hamiltonian operator at the initial time moment. Such states correspond to the Gaussian functionals with the following singular part of the operator R^t:

$$(R * R)(\mathbf{x},\mathbf{k}) \sim -(\mathbf{k}^2 + m^2 + V''(\Phi^t_c(\mathbf{x}))) \tag{73}$$

Formulas (58) and (73) are colloraries of our condition (72) if and only if the asymptotics is constucted as $|\mathbf{k}| \to \infty$ up to $O(1/|\mathbf{k}|)$ and $O(1/|\mathbf{k}|^2)$ accuracy correspondingly. For general non-stationary solutions, this is correct only for sufficiently small number of the space-time dimensions. On the other hand, for stationary solutions the condition (73) is always correct.

Although we have considered the case of the single scalar field only, the more complicated cases can be studied analogously.

For gauge theories [33], one can consider the Coulomb gauge and write the evolution equation in this gauge. Then the Bogoliubov's procedure of "switching on" the interaction can be applied.

For theories with fermions, one can also consider the ansatz (23), where the operator U contains the boson fields only. Eq.(23) will obey the quantum equation of motion if the state vector Y^t satisfies the fermionic Schrödinger equation with the quadratic Hamiltonian. This is an exactly solvable equation [12].

The obtained condition on the operator R does not depend on counterterms; it depends on the classical action only. Thus, the obtained requirement on R can be written both for renormalizable and non-renormalizable cases.

The pre-exponential factor depends on the one-loop counterterm $H^{(1)}$. This is in agreement with the well-known calculations of the functional integral in quantum field theory by the saddle-point technique: tree Feynman graphs correspond to the classical action, while one-loop graphs correspond to the determinat of fluctuations.

The problem of divergences and renormalization in quantum field theory is usually associated with loop Feynman graphs. We see from the obtained results that renormalization is very important even at the tree level, i.e. in the classical field theory. It is the problem of divergences and renormalization that leads us to the conditions on the solution to eq.(60). This equation is classical, not quantum, because it does not depend on the counterterms.

We have also seen that the pre-exponential factor in eq.(57) can be divergent, because the counterterm $H^{(1)}$ compensates the infinite phase of c^t, while the divergence in $|c^t|$ which arises from the combination of determinants (43) cannot be removed. This observation can be interpretted as follows. It is known that the Heisenberg fields $\hat{\varphi}(\mathbf{x},t)$ are well-defined operatoer distributions (i.e. expression $\int dt d\mathbf{x} \hat{\varphi}(\mathbf{x},t) f(\mathbf{x},t)$, where f is a c-number function, determines the operator). However, Schrödinger field operators may not be defined as operator distributions [34,13,35] (i.e. expression $\int d\mathbf{x} \hat{\varphi}(\mathbf{x},t) f(\mathbf{x})$ is not a well-defined operator). This means that the interpretation of the Schrödinger functional as a probability amplitude that the value of field φ is given is not consistent. On the other hand, the notion of the quadratic form entering to the wave functional can be introduced.

We have considered the leading order of the semiclassical approximation only. One can investigate larger orders in analogous way. However, one should take into account the volume divergences corresponding to the Haag theorem [35,34]. This means that the transformation $T_{\Lambda,L}$ should be choosen as a composition of the constructed transformation (50) and the Faddeev transformation [14] which removes the vacuum divergences.

The problem of regularity of the renormalized evolution operator at larger orders of the semiclassical expansion can be reduced to the problem of regularity of the Bogoliubov's

S-matrix corresponding to the interaction which is "swithched" off at the infinity. If the classical solution vanishes, the problem can be solved by using the Bogoliubov-Parasiuk theorem [16,36,37].

We will consider these problems in details in our following publications.

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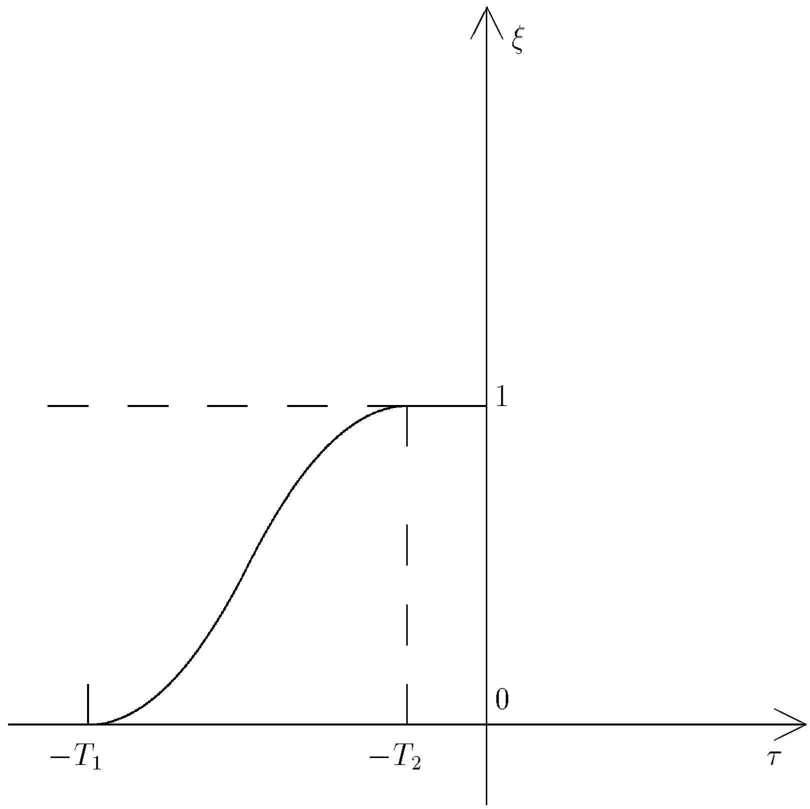

Fig.1